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On the Unique Metro Domination of Bicyclic Graphs

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Given a set of vertices $W = \{w_1, w_2, w_3, ..., w_k\}$ of a connected graph G, the metric representation of a vertex u of G with respect to W is the vector

$$\Gamma(u/W) = (d(w_1, u), d(w_2, u), ..., d(w_k, u))$$

where $d(w_i, u)$, $i \in \{1, 2, \ldots, k\}$ denotes the shortest distance between u and w_i. The set W is said to be a *resolving set* for G, if $\Gamma(u/W) \neq \Gamma(v/W)$, for every $u, v \in V - W$. The minimum cardinality of any resolving set for G is the *metric dimension* of G. A dominating set which resolves a graph G is called a metro dominating set. Further, if $|N(u) \cap W| = 1$ for every vertex $u \in V - W$, then the metro dominating set W of a graph G is called a unique metro dominating set (in short an UMD-set).The minimum of the cardinalities of UMD-sets of G is called unique metro domination number of G denoted by $\gamma_{\mu\beta}(G)$. In this paper, a class of Bicyclic graphs, are studied.

1 Introduction

In the Aerospace industry, navigation and routes needs to be planned in a optimized way so that it is cost effective. One of the ways to reduce costs is optimizing fuel burn. For an airline, this needs to be planned based on the number of available aircraft at a particular location/hanger, availability of pilots, distance to fuelling airports and so on. In order to achieve this optimization, take of location (Node or set of nodes) based on the various input parameters is advantageous. Small resolving sets can also provide an elegant solution to source localization[1].

Let G be a connected graph with the vertex set $V(G)$. The distance $d(u, w)$ between two vertices $u, w \in V(G)$ is the length of a shortest path between them. For a vertex $w \in V(G)$, $N(w)$ denotes the set of all vertices adjacent to w and is called open neighborhood of w. Similarly, the closed neighborhood of w is defined as $N[w] = N(w) \cup \{w\}$. Given an ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the metric representation of a vertex u in G with respect to W is the k-vector denoted by $\Gamma(u/W) = (d(w_1, u), d(w_2, u), ...d(w_k, u))$. The set W is called a *resolving set* of G if $\Gamma(u/W) \neq \Gamma(v/W)$ for every $u, v \in V - W$, or equivalently, if every vertex of G is uniquely identified by its distances from the vertices of W. A resolving set of minimum cardinality is called a *metric basis* and its cardinality is the *metric dimension* of G [2]. If $\Gamma(u/W) = (d_1, d_2, d_3, \ldots, d_k)$, then $d_1, d_2, d_3, \ldots, d_k$ are called components of the code of u generated by W and in particular $d_i, 1 \leq i \leq k$, is called i^{th}-component of the code of u generated by W. A dominating set D of a graph $G(V, E)$ is the subset of $V(G)$ having the property that for each vertex $u \in V(G) - D$, there exists a vertex $w \in D$ such that uw is in $E(G)$ [3],[4].

2 Definitions

Definition 2.1 *Let* $D = \{u_1, u_2, \ldots, u_k\}$ *be an ordered dominating set of* G *and let* w *be a vertex of* G. *The representation* $\Gamma(w/D)$ *of* w *with respect to* D *is the k-tuple* $((d(w, u_1), (d(w, u_2), \ldots, (d(w, u_k))$. *If distinct vertices of* G *have distinct co-ordinates or representation with respect to* D, *then* D *is called a metro dominating set of* G *or simply an MD-set. A dominating set* D *is called minimal if none of its proper subsets is a dominating set. The minimum of cardinalities of minimal MD sets of G is called the lower metro domination number or simply the metro domination number of* G, *denoted by* $\gamma_\beta(G)$ *[7].*

Definition 2.2 *A metro dominating set* D *of a graph* $G(V,E)$ *is a unique metro dominating set (in short an UMD set) if,*

$$|N(u)\cap D|=1$$

for each vertex $u\in V(G)-D$. The minimum of cardinalities of minimal unique metro dominating sets of G is called unique metro domination number denoted by $\gamma_{\mu\beta}(G)$ [6],[10].

3 Bicyclic Graphs

Consider three paths P_a, P_b and P_c where $a+b+c=n$. Join the end vertices of P_b to the end vertices of P_a and P_c. This is called a bicyclic graph denoted by $C^{a,b,c}$. The vertices of P_a are denoted by $v_{1,i}, 1\le i\le a$, the vertices of P_b are denoted by $v_{2,i}, 1\le i\le b$ and the vertices of P_c are denoted by $v_{3,i}, 1\le i\le c$. The graph of n vertices obtained is in fig 1

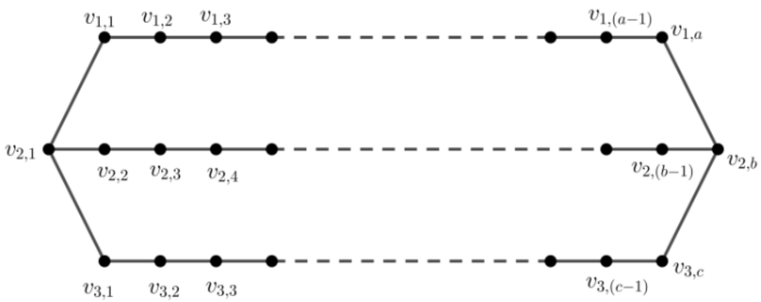


Figure 1: $C^{a,b,c}$

Let D be a subset of the vertex set of G. Let $u,v\in D$ be such that $u-w_1-w_2\dots w_j-v$ is the shortest path between u and v. If none of $w_i, 1\le i\le j$ are in D, i.e., u,v are the only vertices of D in this path, then u and v are neighboring vertices of D and $w_1-w_2-\dots-w_j$ is called a gap between u and v. Further j is called order or length of the graph [8],[9].

In a cycle or path the gaps of D are of length 0,1 or 2, where D is a dominating set. A vertex of D can dominate atmost 2 vertices. Hence we get $|D|\ge\frac{1}{2}(|V|-|D|)\,|D|\ge\frac{|V|}{3}$. This leads to,

Lemma 3.1 *If D is a dominating set then* $|D|\ge\frac{n}{3}$.

In the bicyclic graph $C^{a,b,c}$, if the vertex $v_{2,1}$ and $v_{2,b}$ are in D and all other vertices of D form gaps of order 2, then $2(|D|-2)=|V|-|D|-63|D|=|V|-2\,|D|=\frac{|V|-2}{3}$. This leads to

Lemma 3.2 *If* $v_{2,1},v_{2,b}\in D$ *and all other vertices of* D *form gaps of order 2 then* $|D|=\frac{|V|-2}{3}=\frac{n-2}{3}$

It is observed that $N(v_{2,1})=\{v_{1,1},v_{2,2},v_{3,1}\}$ and $N(v_{2,b})=\{v_{1,a},v_{2,(b-1)},v_{3,c}\}$. If $v_{2,1}$ and $v_{2,b}$ are in D, then P_a, will have $a-2$ vertices to be dominated. Similarly P_c will have c-2 vertices and P_b will have b-4 vertices to be dominated. If all gaps of D are of order 2, then $a-2\equiv 0(mod\ 3)$, $b-4\equiv 0(mod\ 3)$ and $c-2\equiv 0(mod\ 3)$. Further

$$|D|=\frac{a-2}{3}+\frac{b-4}{3}+\frac{c-2}{3}+2=\frac{n-2}{3}.$$

This leads to

Lemma 3.3 *If* $a \equiv 2(mod\ 3)$, $b \equiv 1(mod\ 3)$, $c \equiv 2(mod\ 3)$ *and* $v_{2,1}, v_{2,b} \in D$ *then* $\gamma(C^{a,b,c}) = \dfrac{n-2}{3}$.

In all other cases, $\gamma(C^{a,b,c}) > \dfrac{n-2}{3}$. In all such cases there will be gaps of order 1 or 0.

Observation 3.4

1. If $n \equiv 0(mod\ 3)$, then $\gamma(P_n) = \dfrac{n}{3}$, $\gamma(C_n) = \dfrac{n}{3}$ and all gaps of D are of order 2.
2. If $n \equiv 1(mod\ 3)$, then $\gamma(P_n) = \dfrac{n+2}{3}$, $\gamma(C_n) = \dfrac{n+2}{3}$ and there will be atleast one gap of order 0 or, 1 or more gaps of order1.
3. If $n \equiv 2(mod\ 3)$, then $\gamma(P_n) = \dfrac{n+1}{3}$ and $\gamma(C_n) = \dfrac{n+1}{3}$
4. In P_n or C_n, suppose v_i, v_{i+1} and v_{i+4} are in D forming a gap of order 0 and another gap of order 2. By deleting v_{i+1} from D and adding v_{i+2} to D we get two gaps of order 1
5. In $C^{a,b,c}$, we find three cycles, C_{a+c+2}, C_{a+b} and C_{b+c}. The cycle C_{a+c+2} contains all vertices of P_a and P_c and also $v_{2,1}$ and $v_{2,b}$. The cycle C_{a+b} contains all vertices of P_a and P_b. Similarly C_{b+c} contains all vertices of P_b and P_c.
6. The vertex in a gap of order 1, is not dominated uniquely. Hence as far as possible we avoid gaps of order 1.

Suppose $a \equiv 0(mod\ 3), b \equiv 0(mod\ 3)$ and $c \equiv 0(mod\ 3)$. The cycle C_{a+c+2} contians $v_{2,1}$ and $v_{2,b}$. We take $v_{2,1}$ in D. As $v_{2,2}$ is dominated by $v_{2,1}$ and $v_{2,b} \in C_{a+c+2}$, we are left with $b-3$ vertices $v_{2,3}, v_{2,4} \dots v_{2,b-1}$ of P_b. As $b-3 \equiv 0(mod\ 3)$, we have to include $\dfrac{b-3}{3}$ vertices of P_b into D. They are $v_{2,4}, v_{2,7}, v_{2,10}, \dots, v_{2,(b-2)}$. Similarly keeping gaps of order 2, we include $v_{1,3}, v_{1.6}, \dots, v_{1,a}$ and $v_{3,3}, v_{3,6}, \dots, v_{3,c}$ in D. This leaves a gap of order 1 between $v_{1,a}$ and $v_{3,c}$. Further,

$$a + c + 2 \equiv 2(mod\ 3)$$

and hence

$$|D| = \frac{(a+c+2)+1}{3} + \frac{b-3}{3} = \frac{n}{3}$$

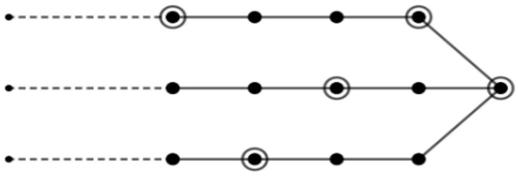
Thus leading to

Lemma 3.5 *If* $a \equiv 0(mod\ 3), b \equiv 0(mod\ 3)$ *and* $c \equiv 0(mod\ 3)$ *then* $\gamma(C^{a,b,c}) = \dfrac{n}{3}$.

Take $a \equiv a_i(mod\ 3), b \equiv b_i(mod\ 3)$ and $c \equiv c_i(mod\ 3)$ where $0 \le a_i, b_i, c_i \le 2$. Then there are 27 cases two of which are given by Lemma 3.3,3.5. Keeping b_i fixed, if the values of a_i and c_i are interchanged, the number of vertices in C_{a+c+2} remains without change. Hence $|D|$ remains the same. For a fixed b_i, the case with $(a_i, b_i, c_i) \equiv (0, b_i, 1)$ is same as the case with $(a_i, b_i, c_i) \equiv (1, b_i, 0)$. As b_i varies, there are 6 cases of which 3 cases with $a_i = 1, c = 0$ and 3 cases with $a_i = 0, c_i = 1$. Similarly when $a_i = 1$ and $c_i = 2$ there are 6 cases and when $a_i = 0$ and $c_i = 2$ there are 6 cases. These 18 cases will have only 9 distinct cases. Hence out of 27 cases 9 cases are dropped. Only 18 distinct cases are for discussion.

Taking $v_{2,1}$, in D, it is seen that $v_{2,4} \in D$ and $v_{1,3k}, k = 1,2,\dots$ are in D. If $v_{1,(a-1)}$ and $v_{1,a}$ are not in D, then $v_{2,b} \in D$. This will imply that $a = 3k+2$. Similarly if $c = 3j+2$ then $v_{2,b} \in D.$ Thus leading to

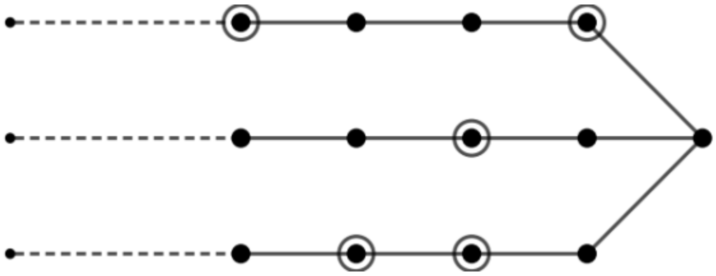
Lemma 3.6 *If* $v_{2,1} \in D$ *and* $a \equiv 2\,(mod\ 3)$ *or* $c \equiv 2\,(mod\ 3)$, *then* $v_{2,b} \in D$



$(a_i, b_i, c_i) \equiv (0,0,2)$ MD Set
Figure 2:

When $a \equiv 0(mod\ 3)$, $b \equiv 0(mod\ 3)$ and $c \equiv 2(mod\ 3)$, $a+c+2 \equiv 1(mod\ 3)$. Further $v_{2,1}$ and $v_{2,b}$ are in D implies that only $b-4$ vertices of P_b are to be dominated and $b-4 \equiv 2(mod\ 3)$. Hence

$$|D| = \frac{(a+c+2)+2}{3} + \frac{b-4+1}{3} = \frac{n+1}{3}.$$



[b] Modified MD Set D
Figure 3:

However there is a gap of order 1 (refer figure 2) between $v_{2,(b-2)}$ and $v_{2,b}$ which can be removed from D.

If $v_{3,(c-1)}$ is included in D and $v_{2,b}$ is removed then the gap of order 1 is removed from D and there will be a gap of order 0 between $v_{3,(c-2)}$ and $v_{3,(c-1)}$ figure 3. Further, $|D|$ is unaltered. This leads to

Lemma 3.7 *If* $a \equiv 0(mod\ 3)$, $b \equiv 0(mod\ 3)$ *and* $c \equiv 2(mod\ 3)$, *then* $\gamma(C^{a,b,c}) = \dfrac{n+1}{3}$

Now take $a_i = 1, b_i = 2$ and $c_i = 2$. Vertex $v_{2,1} \in D$ implies that vertex $v_{2,b} \in D$. Also $a+c+2 \equiv 2(mod 3)$. Further D contains $v_{2,4}, v_{2,7}, \dots,$ .It contain $v_{2,(b-1)}$ which forms a gap of order 0 between $v_{2,(b-1)}$ and $v_{2,b}$. This gap of 0 forces to include $v_{2,b}$ in P_b (for the purpose of calculation of $|D|$) (There are $b-2$ vertices in P_b other than $v_{2,1}$ and $v_{2,b}$. Hence

$$|D| = \frac{a+c+2+1}{3} + \frac{b-2}{3} = \frac{n+1}{3}$$

The gap of order 1 between $v_{1,(a-1)}$ and $v_{2,b}$ figure 4 can be removed without changing $|D|$. If $v_{3,(c-1)}$ is included in D and $v_{2,b}$ is removed from D figure 5 then there is a gap of order 0 between $v_{3,(c-2)}$ and $v_{3,(c-1)}$.

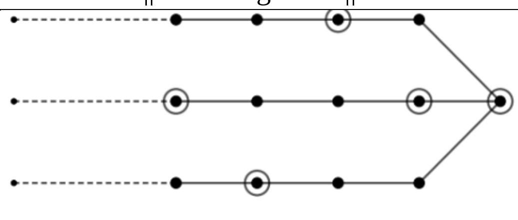


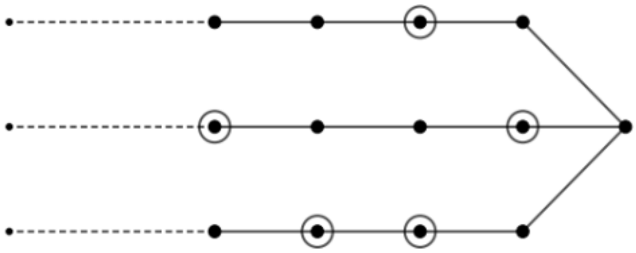
Figure 4:

[width=9 cm]fig3bm

Then no gaps of order 1 will be there. Further $|D|$ remains the same. This leads to

Lemma 3.8 *If* $a \equiv 1 (mod\ 3), b \equiv 2\ (mod\ 3)$ *and* $c \equiv 2\ (mod\ 3)$, *then* $\gamma(C^{a,b,c}) = \frac{n+1}{3}$.

When $v_{2,1}$ and $v_{2,b}$ are in D, there will be $b-4$ vertices of P_b left to be dominated. If $a \not\equiv 2(mod\ 3)$ and $c \not\equiv 2(mod\ 3)$, but $b \equiv 1(mod\ 3)$ then $v_{2,1}, v_{2,4}, v_{2,7} \ldots, \in D$. Hence $v_{2,b} \in D$.



Modified MD Set D

Figure 5:

Take the case when $b \equiv 1(mod\ 3)$, $a \not\equiv 2(mod\ 3)$ and $c \not\equiv 2(mod\ 3)$, say $(a_i, b_i, c_i) \equiv (0,1,1)$. Then $a + c + 2 \equiv 0(mod\ 3)$. Now $b \equiv 1(mod\ 3)$ will imply $\frac{b+2}{3}$ vertices, $v_{2,1}, v_{2,4}, \ldots, v_{2,b} \in D$. Further as $a + b + 2 \equiv 0(mod\ 3)$, there are $\frac{a+c+2}{3}$ vertices of C_{a+c+2} in D. However $v_{2,1}$ is counted twice, in P_b and again in C_{a+c+2}. Therefore there are $\frac{a+c+2}{3} + \frac{b+2}{3} - 1 = \frac{n+1}{3}$ vertices in D. We observe that $v_{2,b}$ is dominated by $v_{2,a}$. But $v_{2,b}$ is required in D to dominate $v_{2,(b-1)}$. Alternately we can have $v_{2,(b-1)} \in D$ instead of $v_{2,b}$. In any case, we have

Lemma 3.9 *If* $a \equiv 0(mod\ \ 3), b \equiv 1(mod\ \ 3)$ *and* $c \equiv 1(mod\ \ 3)$, *then* $\gamma(C^{a,b,c}) = \frac{n+1}{3}$

Suppose $a \equiv 0(mod\ 3)$, $b \equiv 1(mod\ 3)$ and $c \equiv 0(mod\ 3)$. Vertex $v_{2,b} \in D$. As already observed $v_{2,(b-1)}$ can be in D instead of $v_{2,b}$. Then $a + c + 2 \equiv 2(mod 3)$ implies that C_{a+c+2} will have a gap of order 1. The gap of order 1, may be inserted anywhere. In figure 6 , we have the gap of order 0 between $v_{1,a}$ and $v_{2,b}$

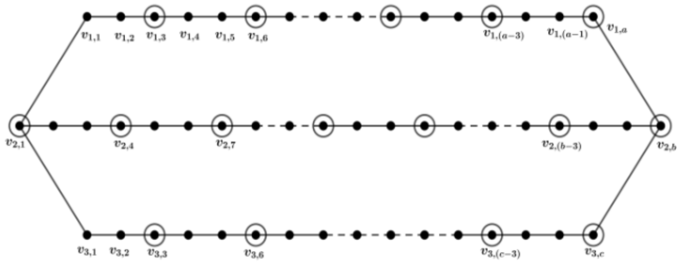


Figure 6:

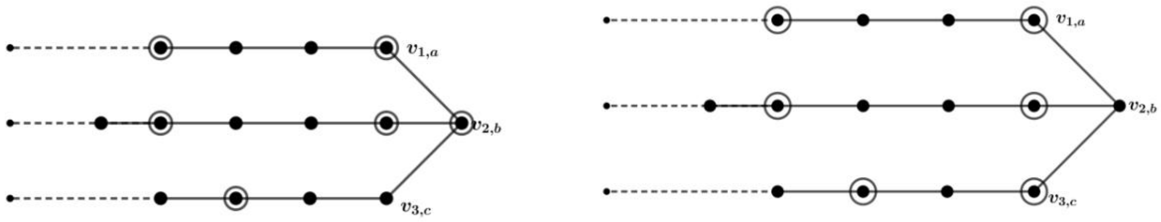
Thus we have

$$|D| = \frac{(a+c+2)+1}{3} + \frac{(b-2)+1}{3} = \frac{n+2}{3}$$

This leads to

Lemma 3.10 *If* $a \equiv 0(mod\ 3), b \equiv 1(mod\ 3)$ *and* $c \equiv 0(mod\ 3)$, *then* $\gamma(C^{a,b,c}) = \frac{n+2}{3}$.

Take the case when $a \equiv 0(mod\ 3), b \equiv 2(mod\ 3)$ and $c \equiv 2(mod\ 3)$. As $a+c+2 \equiv 1(mod\ 3)$, there is a gap of order 0 in C^{a+c+2}. Further $b-3 \equiv 2(mod\ 3)$. Therefore $|D| = \frac{(a+c+2)+2}{3} + \frac{(b-3)+1}{3} = \frac{n+2}{3}$. Gap of order 0 may be converted into 2 gaps of order 1, as seen in figure 7



(A) (B)

Figure 7:

This will lead to the following lemma.

Lemma 3.11 *If* $a \equiv 0(mod\ 3), b \equiv 2(mod\ 3)$ *and* $c \equiv 2(mod\ 3)$, *then* $\gamma(C^{a,b,c}) = \frac{n+2}{3}$.

When $a \equiv 2(mod\ 3)$, $b \equiv 2(mod\ 3)$ and $c \equiv 0(mod\ 3)$, then $v_{2,1}$ and $v_{2,b}$ are in D. As $v_{2,1} \in N(v_{2,1})$ and $v_{2,(b-1)} \in N(v_{2,b})$, remain $b-4$ vertices of P_b. Here $a+b+2 \equiv 1(mod\ 3)$ and $b-4 \equiv 1(mod\ 3)$. Vertices in P_{b-4} are $v_{2,3}, v_{2,4}, v_{2,5}, \ldots, v_{2,(b-2)}$: The vertex $v_{2,(b-4)} \in D$, hence $v_{2,(b-2)} \in D$, creating 2 gaps of order 1. Alternately we may include $v_{2,(b-1)}$ in D instead of $v_{2,(b-2)}$, leaving one gap of order 0. Therefore $|D| = \frac{(a+b+c)+2}{3} + \frac{(b-4)+2}{3} = \frac{n+2}{3}$. This will give us

Lemma 3.12 *If* $a \equiv 2(mod\ 3), b \equiv 2(mod\ 3)$ *and* $c \equiv 0(mod\ 3), \gamma(C^{a,b,c}) = \frac{n+2}{3}$

Going through the remaining cases, we get,

Theorem 3.13 *The domination number of* $C^{a,b,c}$ *is given by*

$$\gamma(C^{a,b,c}) = \begin{cases} \dfrac{n}{3}, & \text{if } a+b+c \equiv 0(mod\ \ 3) \\ \dfrac{n+2}{3}, & \text{if } a+b+c \equiv 1(mod\ \ 3) \text{ and } D \text{ has 2 gaps of order 0} \\ \dfrac{n-1}{3}, & \text{if } a+b+c \equiv 1(mod\ \ 3) \text{ and } D \text{ has no gaps of order 0} \\ \dfrac{n-2}{3}, & \text{if } a+b+c \equiv 2(mod\ \ 3) \text{ if all gaps are of order 2} \\ \dfrac{n+1}{3}, & \text{if } a+b+c \equiv 2(mod\ \ 3) \text{ and } D \text{ has gaps of order other than 2} \end{cases}$$

Suppose there is a gap of order 1 between v_4 and v_6 as in figure 8.

$N(u_5) = \{u_4, u_6\}$ and $D = \{u_1, u_4, u_6, u_9, \ldots\}$. Then $|N(u_5 \cap D| = 2$ and hence u_5 is not uniquely dominated. In such a case we include u_5 into D. Then D is a UMD set as in figure 9.

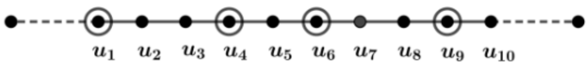


Figure 8

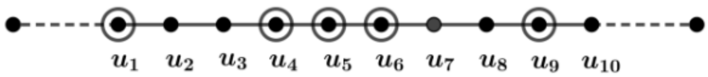


Figure 9

When $a \equiv 0(mod\ \ 3)$, $b \equiv 2(mod\ \ 3)$ and $c \equiv 2(mod 3)$, then D will have 2 gaps of order 0. In figure 12, $a = 6, b = 11$ and $c = 8$. In this case there is a gap of order 0 between $v_{1,6}$ and $v_{2,11}$. There is another gap of order 0 between $v_{2,10}$ and $v_{2,11}$.

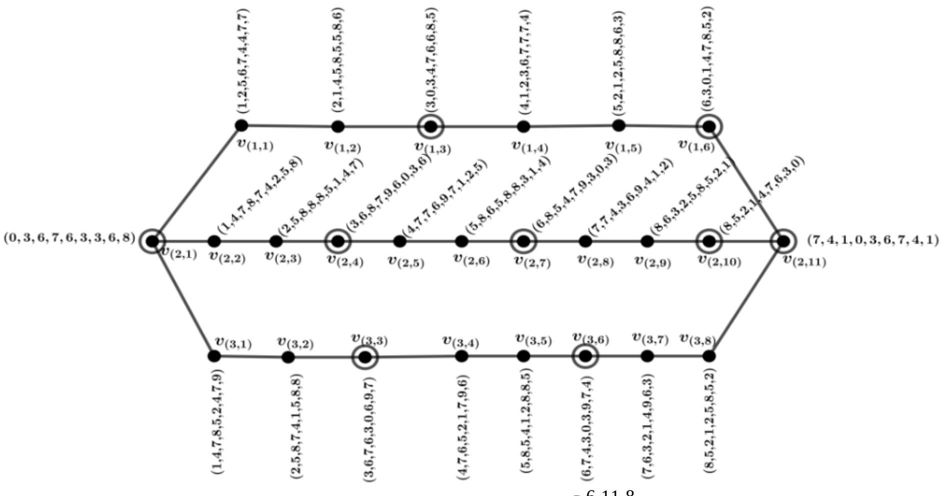


Figure 10: UMD set for $C^{6,11,8}$

In this case $D = \{v_{2,1}, v_{1,3}, v_{1,6}, v_{2,11}, v_{3,6}, v_{3,3}, v_{2,4}, v_{2,7}, v_{2,10}\}$, an ordered set. Here $n = 25$ and

$|D| = \frac{n+2}{3} = 9$. Further D is a UMD set. Vectors assigned to different vertices of $C^{6,11,8}$ are all distinct. Hence $\gamma_{\mu\beta}(C_2^{6,11,8}) = 9$.

If a gap is of order 0 or 2 , then the domination is unique domination. If the gap is of order 1, then domination is not unique.

In the discussion immediately after Lemma 3.5, we defined a_i, b_i and c_i. We now take up the 18 cases based on the triplet (a_i, b_i, c_i)

• Case 1. All gaps are of order 2

In this case the triplets involved are (0,0,1),(2,1,2) and (1,2,1) as seen in figure 11.

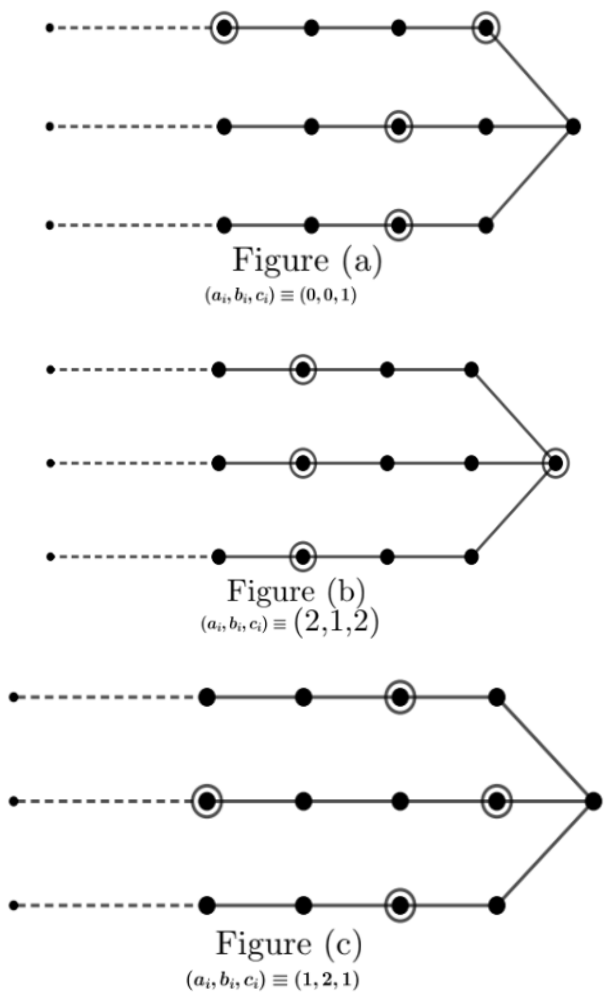


Figure 11:

In all these cases $\gamma(C^{a,b,c}) = \gamma_{\mu\beta}(C^{a,b,c})$

• Case 2. No gaps of order 1 but there are gaps of order 0.

In this case the triplets are (0,0,2),(0,1,0),(0,1,1),(0,1,2),(1,1,1),(0,2,2) and (2,2,2). For all these cases the metro dominating set D is also a unique metro dominating set and hence $\gamma(C^{a,b,c}) = \gamma_{\mu\beta}(C^{a,b,c})$.

• Case 3: The MD set having gaps of order 1, but the adjacent vertices of D with respect to that gap are changed to get a new MD set, as shown in fig 2,3,4, 5. The new MD set D is not having any gap of order 1 . Triplets involved in the cases are (0,0,2) and (1,2,2).

In the above two cases cardinality of the MD set remains same. The modified MD set D has a gap of order 0. Therefore it is a UMD set and hence $\gamma_{\mu\beta}(C^{a,b,c}) = \gamma(C^{a,b,c})$.

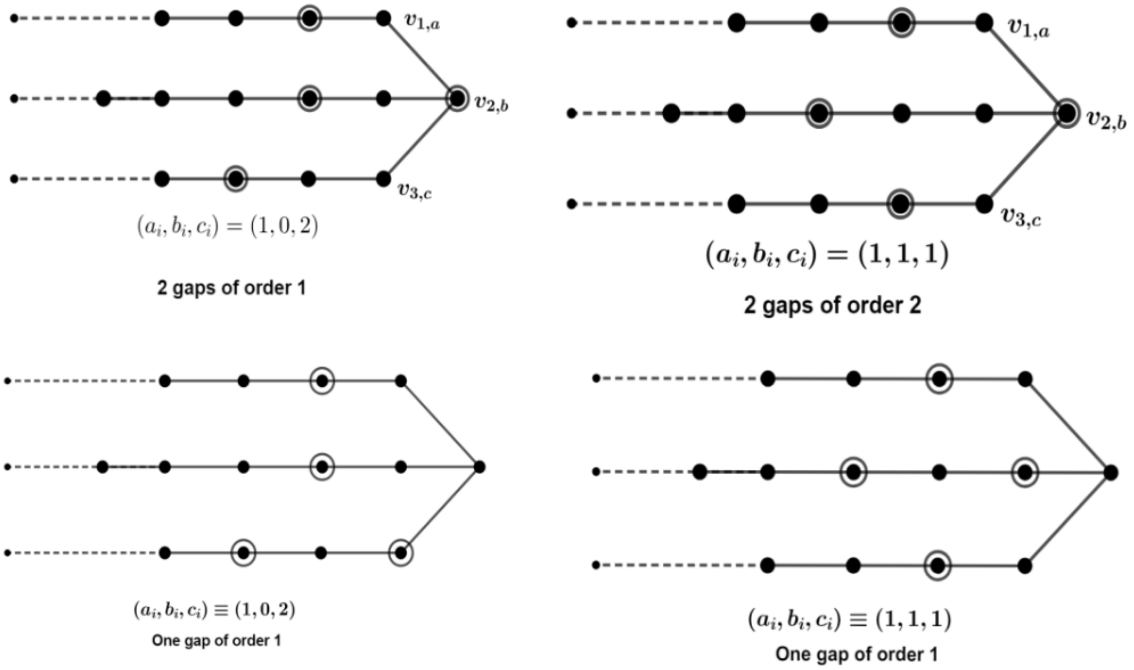


Figure 12:

• Case 4: Metro Dominating set has a gap of order 1. Triplets (a_i, b_i, c_i) involved are (0,0,0),(1,0,2),(2,0,2),(1,1,2),(0,2,0),(0,2,1) and (1,1,1). Two of these (1,0,2) and (1,1,1) have 2 gaps of order 1 which can be modified to a single gap of order 1 as seen in fig. Therefore in all these cases, one vertex is added to the MD set to make it a UMD set. Thus if D is the MD set then we have

Theorem 3.14

$$\gamma_{\mu\beta}(C^{a,b,c}) = \begin{cases} \gamma(C^{a,b,c}), & \text{where D has no gaps of order 1} \\ \gamma(C^{a,b,c}) + 1, & \text{where D has gaps of order 1} \end{cases}$$

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