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A Search on the Homogeneous Cone $x^2+6xy+15y^2=15z^2$

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Abstract: The homogeneous cone represented by the ternary quadratic Diophantine equation $x^2+6xy+15y^2=15z^2$ is studied for finding its non – zero distinct integer solutions. A few interesting properties among the solutions are also exhibited.
Keywords: Homogeneous, Ternary quadratic equation, Integral solutions.

1. Introduction

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2. Notations

- Polygonal number of rank n with size m - $t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$
- Pronic number of rank n - $PR_n=n\,(n+1)$
- Star number of rank n - $S_n=6n^2-6n+1$

3. Method of analysis

The quadratic Diophantine equation with the three unknowns to be solved is given by

$$x^2+6xy+15y^2=15z^2 \tag{1}$$

Substituting,

$$x+3y=u \tag{2}$$

in (1), we get

$$u^2+6y^2=15z^2 \tag{3}$$

Different ways of solving (3) for u, y and z are presented below. Knowing the values of u and y, the corresponding values of x are obtained from (2).

Way 1:
Assume

$$z(a,b)=a^2+6b^2 \tag{4}$$

Write 15 as

$$15=\left(3+i\sqrt{6}\right)\left(3-i\sqrt{6}\right) \tag{5}$$

Using (4) and (5) in (2) and applying the method of factorization define

$$\left(u+i\sqrt{6}y\right)=\left(a+i\sqrt{6}b\right)^2\left(3+i\sqrt{6}\right)$$

From which, on equating the real and imaginary parts, one obtains

$$u=3a^2-18b^2-12ab \tag{6}$$

$$y(a,b) = a^2 - 6b^2 + 6ab \quad (7)$$

In view of (2), we have

$$x(a,b) = -30ab \quad (8)$$

Thus, (4), (7) and (8) represent the integer solutions to (1).

Properties:

1. $x(a,1) + 30\,Pr_a - 30t_{4,a} = 0$
2. $y(a,1) + 5t_{4,a} - 6\,Pr_a + 6 = 0$
3. $y(1,b) + t_{14,b} - Pr_b + t_{4,b} - 1 = 0$
4. $x(a,1) + y(a,1) - 25t_{4,a} + 24\,Pr_a \equiv 0 (\mathrm{mod}\,14)$
5. $x(1,b) + y(1,b) + S_b \equiv 0 (\mathrm{mod}\,14)$

Note 1:

One may also represent 15 as

$$15 = \frac{(9 + i7\sqrt{6})(9 - i7\sqrt{6})}{5^2}$$

After performing a few calculations as above, the corresponding values of x, y and z are given by

$$x(A,B) = -60A^2 + 360B^2 + 690AB$$

$$y(A,B) = 35A^2 - 210B^2 - 90AB$$

$$z(A,B) = 25A^2 + 150B^2$$

Properties:

1. $x(A,1) + t_{122,A} - 631\,Pr_A + 631t_{4,A} - 360 = 0$
2. $x(1,B) - t_{722,B} - 1049\,Pr_B + 1049t_{4,B} + 60 = 0$
3. $y(A,1) - t_{72,A} + 56\,Pr_A - 56t_{4,A} \equiv 0 (\mathrm{mod}\,7)$
4. $y(1,B) + t_{422,B} + 299\,Pr_B - 299t_{4,B} \equiv 0 (\mathrm{mod}\,5)$
5. $x(A,1) + y(A,1) + t_{52,A} - 576\,Pr_A + 576t_{4,A} - 150 = 0$

Way 2:

(3) is written as

$$u^2 + 6y^2 = 15z^2 * 1 \quad (9)$$

Assume

$$1 = \frac{(1 + i2\sqrt{6})(1 - i2\sqrt{6})}{5^2} \quad (10)$$

Substituting (4), (5) and (10) in (9) and employing the method of factorization, define

$$(u + i\sqrt{6}y) = (3 + i\sqrt{6})(a + i\sqrt{6}b)^2 * \frac{(1 + 2i\sqrt{6})}{5}$$

On equating the real and imaginary parts, we get

$$u = \frac{1}{5}(-9a^2 + 54b^2 - 84ab) \quad (11)$$

$$y = \frac{1}{5}(7a^2 - 42b^2 - 18ab) \quad (12)$$

As our interest is on finding integer solutions, replacing a by 5A and b by 5B in (4), (11) and (12), it is seen that

$$u = -45A^2 + 270B^2 - 420AB$$

$$y(A,B) = 35A^2 - 210B^2 - 90AB \quad (13)$$

$$z(A,B) = 25A^2 + 150B^2 \quad (14)$$

In view of (2), one obtains

$$x(A,B) = -150A^2 + 900B^2 - 150AB \tag{15}$$

Thus, (13), (14) and (15) represent the integer solutions to (1).

Properties:

1. $x(A,1) + t_{302,A} + 299\,Pr_A - 299t_{4,A} - 900 = 0$
2. $x(1,B) - t_{1800,B} - 749\,Pr_B + 749t_{4,B} + 150 = 0$
3. $y(A,1) - t_{72,A} + 56\,Pr_A - 56t_{4,A} \equiv 0 (\mathrm{mod}\,3)$
4. $y(1,B) + t_{422,B} + 299\,Pr_B - 299t_{4,B} \equiv 0 (\mathrm{mod}\,7)$
5. $x(A,1) + y(A,1) + t_{232,A} + 354\,Pr_A - 354t_{4,A} - 690 = 0$

Note 2:

The representations of (15) and (1) in (9) may also be considered as follows:

(i) $15 = (3+i\sqrt{6})(3-i\sqrt{6}),\ 1 = \dfrac{(5+i\sqrt{6})(5-i\sqrt{6})}{49}$

(ii) $15 = \dfrac{(9+i7\sqrt{6})(9-i7\sqrt{6})}{25},\ 1 = \dfrac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{25}$

(iii) $15 = \dfrac{(9+i7\sqrt{6})(9-i7\sqrt{6})}{25},\ 1 = \dfrac{(5+i\sqrt{6})(5-i\sqrt{6})}{49}$

Employing the procedure as above for each of this representations, the corresponding integer solutions to (1) thus obtained are presented below.

Solutions obtained from (i):

$$x(A,B) = -7(30A^2 + 180B^2 + 150AB)$$
$$y(A,B) = 7(11A^2 - 66B^2 + 6AB)$$
$$z(A,B) = 49(A^2 + 6B^2)$$

Solutions obtained from (ii):

$$x(a,b) = -6a^2 + 3b^2 + 6ab$$
$$y(a,b) = a^2 - 6b^2 - 6ab$$
$$z(a,b) = a^2 + 6b^2$$

Solutions obtained from (iii):

$$x(A,B) = 35(-198A^2 + 1188B^2 - 402AB)$$
$$y(A,B) = 35(53A^2 - 318B^2 - 78AB)$$
$$z(A,B) = 35^2(A^2 + 6B^2)$$

Way 3:

(3) can be written in the form of ratio as

$$\frac{u+3z}{6(z-y)} = \frac{z+y}{u-3z} = \frac{\alpha}{\beta},\ \beta \neq 0 \tag{16}$$

Which is equivalent to the system of double equations

$$\left.\begin{array}{l} \beta u + 6\alpha y + (-6\alpha + 3\beta) z = 0 \\ -\alpha u + \beta y + (3\alpha + \beta) z = 0 \end{array}\right\} \tag{17}$$

Solving (17) by method of cross multiplication we get,

$$u = 18\alpha^2 - 3\beta^2 + 12\alpha\beta$$

$$\left.\begin{array}{l} y(\alpha,\beta) = 6\alpha^2 - \beta^2 - 6\alpha\beta \\ z(\alpha,\beta) = 6\alpha^2 + \beta^2 \end{array}\right\} \tag{18}$$

Using (2), we have

$$x(\alpha,\beta) = 30\alpha\beta \tag{19}$$

Thus, (18) and (19) represent the integer solutions to (1).

Properties:

$1.x(\alpha,1)-30Pr_{\alpha}+30t_{4,\alpha}=0$

$2.y(\alpha,1)-t_{14,\alpha}+Pr_{\alpha}-t_{4,\alpha}+1=0$

$3.y(1,\beta)+t_{4,\beta}+6Pr_{\beta}-6t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$

$4.x(\alpha,1)+y(\alpha,1)-t_{14,\alpha}-29Pr_{\alpha}+29t_{4,\alpha}+1=0$

$5.x(1,\beta)+y(1,\beta)-24Pr_{\beta}+25t_{4,\beta}\equiv 0(\mathrm{mod}\,6)$

Case 2:

(3) is written in the form of ratio as

$$\frac{u+3z}{2(z-y)}=\frac{3(z+y)}{u-3z}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equations

$$\left.\begin{aligned}&\beta u+2\alpha y+(3\beta-2\alpha)=0\\&-\alpha u+3\beta y+(3\beta+3\alpha)z=0\end{aligned}\right\}\qquad(20)$$

Solving (20) by method of cross multiplication, we get

$$u=6\alpha^2-9\beta^2+12\alpha\beta$$

$$\left.\begin{aligned}&y(\alpha,\beta)=2\alpha^2-3\beta^2-6\alpha\beta\\&z(\alpha,\beta)=2\alpha^2+3\beta^2\end{aligned}\right\}\qquad(21)$$

Using (2),

$$x(\alpha,\beta)=30\,\alpha\beta\qquad(22)$$

Thus, (21) and (22) represent the integer solutions to (1).

Properties:

$1.y(\alpha,1)-t_{6,\alpha}+5Pr_{\alpha}-5t_{4,\alpha}\equiv 0(\mathrm{mod}\,3)$

$2.y(1,\beta)+t_{8,,\beta}+8Pr_{\beta}-8t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$

$3.x(\alpha,1)+y(\alpha,1)-t_{6,\alpha}-25Pr_{\alpha}+25t_{4,\alpha}+3=0$

$4.x(1,\beta)+y(1,\beta)+27t_{4,\beta}-24Pr_{\beta}-2=0$

$5.x(\alpha,1)-y(\alpha,1)+t_{6,\alpha}-35Pr_{\alpha}+35t_{4,\alpha}-3=0$

Case 3:

(3) is written in the form of ratio as

$$\frac{u-3z}{3(z-y)}=\frac{2(z+y)}{u+3z}=\frac{\alpha}{\beta},\beta\neq 0$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned}&\beta u+3\alpha y+(-3\alpha-3\beta)z=0\\&-\alpha u+2\beta y+(-3\alpha+2\beta)z=0\end{aligned}\right\}\qquad(23)$$

Solving (23) by method of cross multiplication,we get

$$u=-9\alpha^2+6\beta^2+12\alpha\beta$$

$$\left.\begin{aligned}&y(\alpha,\beta)=3\alpha^2-2\beta^2+6\alpha\beta\\&z(\alpha,\beta)=3\alpha^2+2\beta^2\end{aligned}\right\}\qquad(24)$$

Using (2),

$$x=-18\alpha^2+12\beta^2-6\alpha\beta\qquad(25)$$

Thus, (24) and (25) represent the integer solutions to (1).

Properties:

$1.x(\alpha,1)+t_{38,\alpha}+23Pr_{\alpha}-23t_{4,\alpha}\equiv 0(\mathrm{mod}\,12)$

$2.x(1,\beta)-t_{26,\beta}-5Pr_{\beta}+5t_{4,\beta}\equiv 0(\mathrm{mod}\,9)$

$3.y(\alpha,1)-t_{8,\alpha}-8Pr_{\alpha}+8t_{4,\alpha}+2=0$

$4.y(1,\beta)+t_{6,\beta}-5Pr_{\beta}+5t_{4,\beta}-3=0$

$5.x(\alpha,1)+y(\alpha,1)+15t_{4,\alpha}-10=0$

Case 4:

(3) is written in the form of ratio as

$$\frac{u+3z}{z-y}=\frac{6(z+y)}{u-3z}=\frac{\alpha}{\beta},\beta\neq 0$$

Which is equivalent to the system of double equations

$$\left.\begin{aligned}\beta u+\alpha y+(-\alpha+3\beta)z=0\\ -\alpha u+6\beta y+(3\alpha+6\beta)z=0\end{aligned}\right\} \quad (26)$$

Solving (26) by method of cross multiplication, we get

$$u=3\alpha^2-18\beta^2+12\alpha\beta$$

$$\left.\begin{aligned}y(\alpha,\beta)=\alpha^2-6\beta^2-6\alpha\beta\\ z(\alpha,\beta)=\alpha^2+6\beta^2\end{aligned}\right\} \quad (27)$$

Using (2),

$$x(\alpha,\beta)=30\alpha\beta \quad (28)$$

Thus, (27) and (28) represent the integer solutions to (1).

Properties:

$1.z(\alpha,1)-t_{4,\alpha}-6=0$

$2.y(\alpha,1)-7t_{4,\alpha}+6Pr_{\alpha}\equiv 0(mod3)$

$3.z(1,\beta)-6t_{4,\alpha}-1=0$

$4.y(1,\beta)+t_{14,\beta}+11Pr_{\beta}-11t_{4,\beta}-1=0$

$5.y(\alpha,1)+z(\alpha,1)+6Pr_{\alpha}-8t_{4,\alpha}=0$

Case 5:

(3) is written in the form of ratio as

$$\frac{u-3z}{z-y}=\frac{6(z+y)}{u+3z}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equations

$$\left.\begin{aligned}\beta u+\alpha y+(-\alpha-3\beta)z=0\\ -\alpha u+6\beta y+(-3\alpha+6\beta)z=0\end{aligned}\right\} \quad (29)$$

Solving (29) by method of cross multiplication, we get

$$u=-3\alpha^2+18\beta^2+12\alpha\beta$$

$$\left.\begin{aligned}y(\alpha,\beta)=\alpha^2-6\beta^2+6\alpha\beta\\ z(\alpha,\beta)=\alpha^2+6\beta^2\end{aligned}\right\} \quad (30)$$

Using (2),

$$x(\alpha,\beta)=-6\alpha^2+36\beta^2-6\alpha\beta \quad (31)$$

Thus, (30) and (31) represent the integer solutions to (1).

Properties:

$1.x(\alpha,1)+t_{14,\alpha}+11Pr_{\alpha}-11t_{4,\alpha}\equiv 0(mod4)$

$2.x(1,\beta)-t_{74,\beta}-29Pr_{\beta}+29t_{4,\beta}\equiv 0(mod3)$

$3.y(\alpha,1)-t_{4,\alpha}-6Pr_{\alpha}+6t_{4,\alpha}+6=0$

$4.y(1,\beta)+t_{14,\beta}-Pr_{\beta}+t_{4,\beta}-1=0$

$5.x(\alpha,1)+y(\alpha,1)+5t_{4,\alpha}-30=0$

Way 4:

Introducing the linear transformations

$$z = X + 6T,\ y = X + 15T,\ u = 3U \tag{32}$$

in (3), it is written as

$$X^2 = 90T^2 + U^2 \tag{33}$$

which is satisfied by

$$T = 2rs,\ U = 90r^2 - s^2,\ X = 90r^2 + s^2$$

In view of (32) and (2), the corresponding integer solutions to (1) are given by

$$x = -6s^2 - 90rs$$

$$y = 90r^2 + s^2 + 30rs$$

$$z = 90r^2 + s^2 + 12rs$$

Also, (33) can be expressed as the system of double equations as presented below in Table 1:

Table 1: System of Double Equations

System	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$X+U$	T^2	$3T^2$	$5T^2$	$9T^2$	$15T^2$	$45T^2$	90T	45T	30T
$X-U$	90	30	18	10	6	2	T	2T	3T

System	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
$X+U$	18T	15T	10T	9T	6T	5T	3T	2T	T
$X-U$	5T	6T	9T	10T	15T	18T	30T	45T	90T

Solving each of the above system of equations, the values of X, U and T are obtained.

In view of (32) and (2), the corresponding integer solutions to (1) are obtained. For simplicity, we present below the corresponding solutions in Table 2:

Table 2: Solutions

System	**x**	**y**	**z**
1	$-270-90k$	$2k^2+30k+45$	$2k^2+12k+45$
2	$-90-90k$	$6k^2+30k+15$	$6k^2+12k+15$
3	$-54-90k$	$10k^2+30k+9$	$10k^2+12k+9$
4	$-30-90k$	$18k^2+30k+5$	$18k^2+12k+5$
5	$-18-90k$	$30k^2+30k+3$	$30k^2+12k+3$
6	$-6-90k$	$90k^2+30k+1$	$90k^2+12k+1$
7	$-96k$	121k	103k
8	$-102k$	77k	59k
9	$-108k$	63k	45k
10	$-120k$	53k	35k
11	$-126k$	51k	33k
12	$-144k$	49k	31k
13	$-150k$	49k	31k
14	$-180k$	51k	**33k**
15	$-198k$	53k	**35k**
16	$-270k$	63k	45k
17	$-360k$	77k	59k
18	$-630k$	121k	103k

4. Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties of the homogeneous ternary quadratic equation represented by $x^2 + 6xy + 15y^2 = 15z^2$.

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